FINITE SETS: Cardinality & Functions between Finite Sets

(summary of results from Chapters 10 & 11)

From previous chapters:

the composition of two injective functions is injective, and the
the composition of two surjective functions is surjective -- > and hence the same for two bijective functions.
Also recall that a function $f: X \to Y$ is bijective if and only if it has an inverse $f^{-1}: Y \to X$.

- Notation: $\mathbb{N}_n \stackrel{\text{def}}{=} \{1,2,\ldots,n\}$. Ex: $\mathbb{N}_{10} = \{1,2,3,4,5,6,7,8,9,10\}$
- Def: The **<u>cardinality</u>** of a set X is denoted by |X| and its mathematical definition is:
  - $|\phi| \stackrel{\text{def}}{=} 0$
  - if X is a non-empty finite set, by definition $|\boldsymbol{X}| = n \in \mathbb{Z}^+$ iff $\exists\, a\ bijection\ f: \mathbb{N}_n \to X$.
- Def: A set X is **<u>finite</u>** $\Leftrightarrow$ $\exists$ an integer $n \geq 0$ such that $|X|=n$

  $\Leftrightarrow$ either $X = \phi$, or $\exists\, n \in \mathbb{Z}^+$ such that $\exists$ a bijection $f: \mathbb{N}_n \to X$
- Def: A set X is **<u>infinite</u>** (it is not finite) $\Leftrightarrow$ there exist no bijection $f: \mathbb{N}_n \to X$, for any $n \in \mathbb{Z}^+$

<u>Lemma 10.1.4</u> (proof in Ch 11, by induction on n) ***: If $f: \mathbb{N}_m \longrightarrow \mathbb{N}_n$ is an injection, then $m \leq n$.***

<u>Corollary</u>: **If $f: \mathbb{N}_m \longrightarrow \mathbb{N}_n$ is a bijection, then $m = n$.**

<u>Proposition 10.1.3</u> **If $f: \mathbb{N}_m \longrightarrow X$, $g: \mathbb{N}_n \longrightarrow X$ are two bijections with the same codomain X, then $m = n$.**

(This proposition shows that the mathematical definition of |X| is well-defined.)

<u>Other Corollaries of Lemma 10.1.4 (understand and be able to use in problems):</u>

<u>Corollary 11.1.1:</u> Let X, Y be two non-empty finite sets. If there exists an injection $\boldsymbol{f}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, then $|X| \leq |Y|$

<u>Theorem 11.1.2</u> (**PIGEONHOLE PRINCIPLE**): Let X, Y be two non-empty finite sets.
Suppose $\boldsymbol{f}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is a function, and that $|X| > |Y|$. Then *f* is NOT injective.

<u>Theorem 11.1.6</u> Suppose $\boldsymbol{f}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ is a function between non-empty finite sets, and $|\boldsymbol{X}| < |\boldsymbol{Y}|$.
Then f is NOT surjective.

<u>Theorem 11.1.7:</u> If you have a function $\boldsymbol{f}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ between 2 **<u>finite</u>**, non-empty sets, of **<u>equal cardinality</u>**, then:

f is bijective $\Leftrightarrow$ f is surjective $\Leftrightarrow$ f is injective

<u>Proposition 11.1.4</u> Suppose $\boldsymbol{f}: \boldsymbol{X} \longrightarrow \mathbb{N}_n$ is an injection. Then X is a finite set and $|X| \leq n$.

<u>Corollary 11.1.5:</u> Suppose $\mathbf{X} \subseteq \mathbf{Y}$, and **Y** is a finite set. Then X is also a finite set, and $|X| \leq |Y|$.

<u>Counting Principles (Ch 10)</u>: If X, Y are two finite sets, then:

1. <u>Inclusion-exclusion Principle</u>: $|X \cup Y| = |X| + |Y| - |X \cap Y|$
2. <u>Multiplication Principle</u>: $|X \times Y| = |X||Y|$